A Note on the Mechanics of Ancient Gear Systems

F. Sorge*
University of Palermo
Palermo, Italy

Abstract—This paper deals with the mechanical behavior of the gearwheels of the antiquity, which were generally characterized by triangular shaping of the toothing. The engagement of the conjugate profiles is analyzed in detail, calculating the temporal variation of the speed ratio due to the back and forth shifting of the relative instant center, searching for the admissibility conditions of the points of the theoretical contact path, estimating the rattling level because of the small successive tooth collision and ascertaining the energy losses connected with the particular nature of the coupling. A very interesting result is that only one couple of teeth turns out to be active at each time instant and the entire real path may belong to the only access region entirely or to the only recess one entirely or may be split into two separate sub-phases, the one in access and the other in recess, or may straddle both regions. The occurrence of each of these situations depends on the average speed ratio (tooth ratio) and on the assigned allowance between the two wheel axes. It is also found that the speed oscillation is roughly contained in a ±10% range and the efficiency may reach rather high values, despite the presumable crude finishing of the ancient gearwheels due to the rudimentary technology used in the construction of the tooth profiles.

Keywords: triangular tooth gearing, history of mechanics

I Introduction

Though only very few residues from the antiquity machinery are still preserved in some museums scattered over the world, it is legitimate to guess that a relatively advanced construction technology had been achieved and to imagine an extended practical use of many mechanical devices, especially in the Hellenistic, Byzantine and Islamic worlds.

The gearwheel coupling was no doubt a somehow current application and, for example, was largely used for the implementation of astronomical devices, like planetary calculators for the position of many celestial bodies, or astrolabes, or odometers.

One of the most significant find, the Antikythera mechanism (Fig. 1), is a planetary gear system, presumably of the first century B.C., ascribed perhaps to the philosopher Posidonius or to the astronomer Hypparchus of the Academy of Rhodes and used for the calculation of several astronomical positions. It was retrieved at the beginning of the XX century from the Antikythera wreck, which was accidentally discovered thanks to some sponge-divers anchored near the coast of the homonymous island ("Αντίκυθαρα", whose meaning is "in front of Kythera", is a very small Greek island with less than one hundred inhabitants in the sea channel between Crete and the larger island of Kythera).

Many in-depth studies have been carried out on its functionality as a primitive analog computer (for example, see Pastore [1], de Solla Price [2-3], Wright [4] and mind the recent activity of the Antikythera Mechanism Research Project [5]). Reference [1] by Pastore reports an extensive and careful description of this gear system and elucidates its functional characteristics. De Solla Price spent a lot of time in his studies about this mechanism, in order to reconstruct the missing parts starting from the few archeological residuals, and also tried to assemble a complete model, whose copy is now in the National Archaeological Museum of Athens. Wright carried out a wide campaign of X-ray detection of the wheel, continued later on by the Antikythera Mechanism Research Project, which pointed out the shape of equilateral triangles of the toothing unequivocally. Nevertheless, it is sensational that such a profile denounces a less advanced design conception in comparison with the recent find of the gearwheel of Olbia (Sardinia, Italy), which may be ascribed to the genius of Archimedes of Syracuse (third century B.C.) and is then earlier than Antikythera planetary of more than one century, but exhibits the extraordinary characteristic that the tooth profiles are very close to the modern cycloidal shape [1].

It is very probable that many gear systems like the Antikythera mechanism were built during the Hellenistic period. Cicero mentions two other devices of this type in De Re Publica and says that they had been built by Archimedes and one of them was brought to Rome by Marcus Claudius Marcellus, who conquered Syracuse during the Second Punic War. It is believed that the cycloidal gearwheel of Olbia belonged to one of these devices [1]. Cicero's also says that another such device had been built "recently" by a friend of him and thus, this technology was quite spread since the time of Archimedes and the Antikythera orrery was only one exemplar of a widely diffused manufacturing, though skilled hands and complex calculations were needed for this type of construction.

*sorge@dima.unipa.it
It is assumed that the teeth on the one and the other gear have the same aperture angle $2\beta$ and the same height $h$, whence, fixing the tooth numbers, the whole toothing can be designed relying on evident geometrical considerations.

The triangle $OBV$ in the detail of Fig. 1 shows that the tooth height $h$ and the side edges $e$ are calculable as

$$h = R \left[ 1 - \frac{\sin \beta}{\sin \left( \beta + \frac{\pi}{z} \right)} \right] \quad (1)$$

and

$$e = R \frac{\sin \frac{\pi}{z}}{\sin \left( \beta + \frac{\pi}{z} \right)} \quad (2)$$

for each gearwheel, where $R$ and $R - h$ are the tip and root radii and $z$ is the tooth number.

Thus, the assumption of equal heights permits writing

$$\frac{R_2}{R_1} = \frac{1 - \sin \beta}{1 - \frac{\sin \beta}{\sin \left( \beta + \frac{\pi}{z_1} \right)}} \quad (3)$$

and calculating for example $R_2$ once fixed $R_1$, apart from some free scale factor.

The minimum value of the center distance $D$ is $D_{\text{min}} = R_1 + R_2 - h$, but an allowance factor $a$, a little greater than 1, has to be multiplied by $D_{\text{min}}$ necessarily in order to consider some backlash, which was especially needed in the antiquity to let the gear system work, in consideration of the unavoidable manufacturing inaccuracy because of the crude technology of those days. Thus, $D = aD_{\text{min}}$.

After fixing $D$, the ideal angular width of the meshing region is specified by the intersections of the two tip circumferences, of radii $R_1$ and $R_2$:

$$\cos \alpha_{\text{max}} = \frac{D^2 + R_j^2 - R_i^2}{2DR_i} \quad (4)$$

where $i$ and $j$ may stand either for 1 and 2 or for 2 and 1, while $\alpha_i$ indicates the generic angular position of the gearwheel $i$, which is positive or negative for the meshing entrance or exit of both wheels, according to Fig. 1.
III. Kinematics of the gear coupling

Starting from the initial intersection point on the first intersection of the two tip circles, where the two teeth are in contact through the apices, there is a first phase, where the apex of the driven tooth is pushed and slides along the side of the driven tooth, and a second phase, where the driver tooth apex pushes the side of the driven tooth, as far as the ending meshing point, on the second intersection of the tip circles. Such two phases are separated by the matching position of the two profiles, which occurs for \( \alpha_2 = -\alpha_1 \). This matching value will be called \( \alpha_{2m} \) and the two phases will be named of access and recess, though somehow improperly if comparing with the modern gear terminology.

Thus, the access contact locus coincides with the arc of the driven tip circle preceding the matching position \( \alpha_{2m} < 0 \) and the recess contact locus with the arc of the driver tip circle following the matching position \( \alpha_{2m} > 0 \). The instant center of the relative motion is given by the intersection of the normal to the active profile, of the driver wheel in access \( (n_a) \) and of the driven one in recess \( (n_r) \), and the center line. A sudden change of the speed ratio must occur when passing from the matching position of the two teeth.

In dependence on \( \alpha_1 \), it is possible to calculate the driven angle \( \alpha_2 \) and the distance \( v = \sqrt{V_1 V_2} \) between the two vertices by use of specific closure equations, different for the access and the recess.

**A. Access**

Apply the two closure equations

\[
\begin{align*}
R_1 \cos \alpha_1 - v \cos(\alpha_1 + \beta) + R_2 \cos \alpha_1 &= D \\
R_1 \sin \alpha_1 - v \sin(\alpha_1 + \beta) - R_2 \sin \alpha_1 &= 0
\end{align*}
\]

(5) (6)

to get, by elimination of \( v \),

\[
\alpha_2(\alpha_1) = \arcsin\left[ \frac{D \sin(\alpha_1 + \beta) - R_2 \sin \beta}{R_1} \right] - \alpha_1 - \beta
\]

(7)

Then, calculating \( \cos \alpha_2 \) and \( \sin \alpha_2 \) by Eqs. (5-6), squaring and summing, \( \alpha_2 \) is easily eliminated and we get a quadratic equation for \( v \),

\[
v^2 + 2D [D \cos(\alpha_1 + \beta) - R_1 \cos \beta] + \left[ D^2 + R_1^2 - 2DR_1 \cos \alpha_1 - R_2^2 \right] = 0
\]

(8)

where the coefficient of the linear term is equal to twice the positive difference of the projections of \( D \) and \( R_1 \) on the straight line containing the diver profile, while the third term is equal to \( \overline{OB} \overline{V_1}^2 - R_2^2 \) and is negative as \( V_1 \) is inside the driven tip circle. These observations address the choice between the two roots of Eq. (8) towards the plus sign.

\[
v(\alpha_1) = R_1 \cos \beta - D \cos(\alpha_1 + \beta) + \sqrt{\left[ D^2 + R_1^2 - 2DR_1 \cos \alpha_1 - R_2^2 \right]}
\]

(9)

and it may be noticed that the term under the square root is equal to the projection of the peak radius \( R_2 \) on the straight line of the driver profile.

The access phase ends when \( \alpha_2 = -\alpha_1 \) \( (\alpha_1 = \alpha_{2m}) \), whence we get by Eq. (7):
The instant center $C_a$ of the relative motion is found on the center line, tracing the normal $n_a$ to the driver tooth profile, whose slope is equal to $\pi/2 - \alpha_1 - \beta$. Equating the projections of $O_aC_a$ and $O_aV_a$ along the direction parallel to the driver profile, the distance $O_aC_a$ is found to be

$$O_aC_a = \frac{\sqrt{R_1^2 - [D\sin(\alpha_1 - \beta) - R_2 \sin \beta]^2}}{\cos(\alpha_1 + \beta)}$$

whence the speed ratio $\tau = \omega_2/\omega_1 = O_aC_a/O_aV_a = D/O_aC_a - 1$ is obtainable as

$$\tau(\alpha_i) = \frac{D \cos(\alpha_1 + \beta)}{\sqrt{R_1^2 - [D\sin(\alpha_1 - \beta) - R_2 \sin \beta]^2}} - 1$$

**B. Recess**

After passing the matching configuration, the apex of the driver tooth slides on the driven tooth side, whose slope is $\beta - \alpha_2$, and the two closure equations are

$$R_1 \cos \alpha_1 - v \cos(\beta - \alpha_2) + R_2 \cos \alpha_2 = D$$

$$R_1 \sin \alpha_1 - v \sin(\beta - \alpha_2) - R_2 \sin \alpha_2 = 0$$

As it is desired to express all variables as functions of $\alpha_1$, the distance $v$ may be firstly calculated, solving Eqs. (13-14) for $\cos \alpha_2$ and since $\alpha_2$, squaring and summing

$$v^2 - 2R_1 \cos \beta + \left[R_1^2 - (D^2 + R_2^2 - 2DR_1 \cos \alpha_1)\right] = 0$$

where the third term is equal to $R_2^2 - O_aV_1^2$ and is positive, so that both roots of Eq. (15) are positive. Nevertheless, the geometric meaning of Eq. (15) is that, fixing the angular position $\alpha_i$ of the vertex $V_1$, which is inside the circumference of radius $R_2$, one has to find the position of point $V_2$ on this last circumference so that the angle $V_1V_2O_a$ is equal to $\beta$. Two possible positions fulfill this condition, but the one closest to $V_1$ is clearly to be selected, i.e. the lowest root $v$:

$$v(\alpha_i) = R_2 \cos \beta + \sqrt{D^2 + R_2^2 - 2DR_1 \cos \alpha_1} - R_2^2 \sin^2 \beta$$

Since $D^2 + R_2^2 - 2DR_1 \cos \alpha_1 = O_aV_1^2$, since $R_2 \sin \beta$ gives the minimum distance of the centre $O_2$ from the straight line prolonging the side of the driven tooth and since the apex $V_1$ lies on this line, the quartic discriminant under square root of Eq. (16) is certainly positive.

After obtaining $v$, it is possible to get $\alpha_2$ by back calculation

$$\alpha_2(\alpha_1) = \arctan\left[\frac{R_2 \sin \alpha_1}{D - R_2 \cos \alpha_1} - \frac{v \sin \beta}{R_2 - v \cos \beta}\right]$$

and Equation (17) may be observed to be equivalent to stating that the absolute value of $\alpha_2$ is given by the sum of the angles formed by $O_2V_1$ with the center line and with the radius $O_2V_2$.

The instant center $C_a$ of the relative motion during the recess phase is located on the center line, on the intersection with the normal $n_2$ to the driven tooth profile, whose slope is equal to $\pi/2 - \beta + \alpha_2$. Equating the projections of $O_aC_a$ and $O_aV_a$ along the direction parallel to the driven profile, the distance $O_aC_a$ is given by

$$O_aC_a = R_2 \cos(\beta - \alpha_1 - \alpha_2)$$

and the speed ratio $\tau = \omega_2/\omega_1$ is

$$\tau(\alpha_i) = \frac{R_2 \cos(\beta - \alpha_1 - \alpha_2)}{D \cos(\beta - \alpha_1 - \alpha_2)}$$

where $\alpha_2(\alpha_1)$ is given by Eq. (17).

**C. Real meshing conditions**

Of course, the teeth are closely distributed in two regular sequences, on the one and the other wheel, and, after ascertaining that the contact conditions cannot but be described by the analysis of the previous subsections, it is necessary to investigate if all the ideal contact points are admissible and if several meshing couples may be in contact simultaneously.

Choosing a generic contact point on the ideal path, either in access or in recess, the homologous points of the preceding or following tooth couples have angular distances $\pm 2\pi j/\gamma_1$ and $\pm 2\pi j/\gamma_2$ in the driver and driven gearwheels respectively, where $j = 1, 2, \ldots$ Therefore,
drawing the full diagram $\alpha_i(\alpha_j)$ a generic point must be regarded as an admissible contact only if, tracing a straight line with slope $\pi/\zeta$ through such a point, all the other points whose abscissae differ of $\pm 2\pi/\zeta$ from it lie below the diagram, as otherwise there would be interference for some tooth couple.

Since the diagram $\alpha_i(\alpha_j)$ will be shown to exhibit a slight upward concavity nearly everywhere except at the matching position $\alpha_{in}$, where a sudden slope change occurs, the above reasoning leads to exclude all points of the diagram that lie above the prolongation of that particular chord (or else sum of aligned chords) with slope $\pi/\zeta$ and projection $2\pi/\zeta$ on the $\alpha_i$ axis (or sum of projections equal to $2\pi/\zeta$), that is located as much as possible at the very bottom of the main concavity. The admissible contact points, which must lie below this straight line, are thus to be searched, for each ideal point of the diagram, by checking the interference condition of the preceding and following teeth.

The consequence is in practice that only one couple of profile may be in contact at each time instant and, as soon as such profiles detach themselves, two new profiles, either of the following or of the preceding tooth couple, join simultaneously to mesh, either upstream or downstream. This will be better elucidated by showing some practical results from the numerical calculations.

**D. Rattling**

The motion transmission is continuous, but the speed ratio is variable, as the position of the instant center of the relative motion varies back and forth along the center line. Moreover, since the entrance speed ratio is always a little higher than the exit one, because the relative instant center shifts towards the driver wheel center during each partial engagement, a slight impact occurs at the beginning of any mesh phase or sub-phase, as the driven profile has a slightly lower speed before the engagement than after it.

It is supposed that the impact is inelastic, so that, immediately after the conjunction of two profiles, the velocity components of the driven and driven point along the normal to the contact are equal, while immediately before this instant, the driven velocity component is smaller, where the difference is proportional to the speed ratio jump.

The velocity components normal to the active profiles may be obtained as the total velocities of those two points of the virtual rigid planes rotating together with the one and the other wheel, that are located on the intersections, of the parallels to the profile traced from the wheel centers, with the normal to the profile itself.

It will be shown that, for not too large backlash, the meshes phase is entirely in the access region for $\pi/\zeta$ rather larger than 1 and in the recess one for $\pi/\zeta$ rather smaller than 1. In the neighborhood of $\pi/\zeta = 1$, we may observe two sub-phases, the one in access and the other in recess, distant one angular pitch from each other and with a trend to join into a single phase on increasing the backlash. Whatever the tooth ratio may be, the meshing phase tends to straddle the access and recess regions on increasing the backlash, as is also intuitive.

If the whole meshing phase is in the access region, indicating the driver and driven angular speeds with $\omega_i$ and $\omega_d = \tau \omega_i$, the velocity components normal to the active (driver) profile are

$$v_{i1} = \omega_i \, \overline{OC_{1}} \cos(\beta + \alpha_i)$$

$$v_{i1} = \omega_i \, \overline{OC_{1}} \cos(\beta + \alpha_i) = \omega_i \, \overline{OC_{1}} \left\{ D \cos(\beta + \alpha_i) - [D \sin(\beta + \alpha_i) - R_i \sin \beta] \right\}$$

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$$v_{i1} = \omega_i \, \overline{OC_{1}} \cos(\beta + \alpha_i) = \omega_i \, \overline{OC_{1}} \left\{ D \cos(\beta + \alpha_i) - [D \sin(\beta + \alpha_i) - R_i \sin \beta] \right\}$$

indicating the conditions preceding and following the impact with the superscripts − and + respectively and assuming for example that $\omega_i$ is constant, one has $v_{i1} = v_{i1}^+$ and $v_{i1} = v_{i1}^+$, whence $v_{i1} - v_{i1}^- = \tau \omega_1/v_{i1} = v_{i1}^+ - v_{i1}^-$, so that $(v_{i1} - v_{i1}^-)/v_{i1} = 1 - \tau / \tau'$ and, calculating the speed ratios at the beginning and the end of the meshing phase by Eq. (12), one obtains

$$v_{i1} - v_{i1}^- = \frac{D \cos(\beta + \alpha_i)}{\sqrt{R_i^2 - [D \sin(\beta + \alpha_i) - R_i \sin \beta]^2} - 1}$$

$$v_{i1} = \frac{D \cos(\beta + \alpha_i)}{\sqrt{R_i^2 - [D \sin(\beta + \alpha_i) - R_i \sin \beta]^2} - 1}$$

where the subscripts $i$ and $e$ define the initial and ending values of the angle $\alpha_i$.

When the whole meshing occurs in the recess region, the velocity components normal to the active (driver) profile are

$$v_{i1} = \omega_i \, \overline{OC_{1}} \cos(\beta - \alpha_i - \alpha_j)$$

$$v_{i1} = \omega_i \, \overline{OC_{1}} \cos(\beta - \alpha_i - \alpha_j) = \omega_i \left\{ D \cos(\beta - \alpha_i - \alpha_j) - [D \sin(\beta - \alpha_i - \alpha_j) - R_i \sin \beta] \right\}$$

Proceeding as in the previous case, but using now Eq. (19), one gets

$$v_{i1} - v_{i1}^- = \frac{D \cos(\beta - \alpha_i - \alpha_j)}{\cos(\beta - \alpha_i - \alpha_j) - R_i}$$

$$v_{i1} - v_{i1}^- = \frac{D \cos(\beta - \alpha_i - \alpha_j)}{\cos(\beta - \alpha_i - \alpha_j) - R_i}$$

If the meshing begins in the access region and ends in the recess one, which may occur for large allowances, it is possible to use both equations Eqs. (12) and (19),
minding that the initial speed ratio refers to the access and the ending one to the recess:

\[
\frac{v'_{1b} - v'_{2b}}{v_{1a}} = 1 - \frac{R_i \cos(\beta - \alpha_i - \alpha_{1r})}{D \cos(\beta - \alpha_i - \beta - \alpha_{1r} - \alpha_{1s}) - R_i \sin(\beta - \alpha_i - \beta - \alpha_{1r} - \alpha_{1s})} \quad (26)
\]

At last, if the meshing phase splits into two sub-phases, the one in the access region and the other in the recess one, their total width and the distance between each other are both equal to the angular pitch (see Section V). Therefore, each couple of profiles starts the engagement in the access region and meshes for a period shorter than the angular pitch, at whose end the preceding couple engages in the recess region and meshes for the complement angular pitch, until separating simultaneously with the engagement of the new couple of profiles, following the first one, in the access region. For the impact at the beginning of the access sub-phase, Equation (26) may be used, choosing properly \(\tau_i\), (end of the preceding recess sub-phase), while for the starting impact of the recess sub-phase, one has to use an inverse relationship with respect to Eq. (26), minding that \(\tau_i\) refers now to the end of the preceding access sub-phase:

\[
\frac{v'_{1b} - v'_{2b}}{v_{1a}} = 1 - \frac{D \cos(\beta - \alpha_i)}{\sqrt{R_i^2 - [D \sin(\beta + \alpha_i) - R_i \sin \beta]^2} - R_i \cos(\beta - \alpha_i - \beta - \alpha_{1r} - \alpha_{1s})} \quad (27)
\]

Likewise, in the particular case when the engagement straddles both, access and recess regions, in addition to the impact described by Eq. (26), also the impact of the tooth edges at the matching point must be taken into account. All the intermediate points of the driven edge between the two tooth tips bounce on the driver edge, except the inner point, which remains in contact, and the velocity jump is still given by Eq. (27), provided that one replaces \(\alpha_{1r}\) and \(\alpha_{1s}\) with \(\alpha_{in}\), and \(\alpha_{1s}\) and \(\alpha_{2s}\) with \(-\alpha_{on}\).

All these impacts involve relative speed jumps up to the order of 10% and may produce a significant rattle of the gear system if the driving crank has an appreciable angular speed.

IV. Power loss

Apart from the other energy losses, for example in the supports, or because of the air drag, or because of other loss sources, the losses to be ascribed to the tooth meshing are due to the sliding friction and to the tooth impact.

The ideal pressure angle, formed by the normals to the active profile and to the center line, is \(\beta + \alpha_i\) in the access phase and \(\beta - \alpha_i\) in the recess one and is quite larger in comparison with the modern involute toothing.

Owing to the sliding friction, the line of action of the force exerted by the driver tooth on the driven one is rotated of the friction angle \(\varphi = \arctan f\) with respect to the normal, towards the center of the driven wheel, both in the access and in the recess phases (see Fig. 2).

Since only one couple of teeth is active at each time instant, the problem is isostatic and the mutual force is given by the ratio of the driver torque \(M_i\), which is assumed constant, except the instantaneous peaks due to the tooth collisions, and the corresponding arm \(b_1\).

Tracing the parallel to the line of action of the mutual force through the relative instant center as in Fig. 2, the arms \(b_1\) and \(b_2\) are calculable as \(b_1 = O_iC_s \cos(\beta + \alpha_i + \varphi) + V_sC_s \sin \varphi\), \(b_2 = O_iC_s \cos(\beta + \alpha_i + \varphi) + V_sC_s \sin \varphi\) for an access contact, and \(b_1 = O_iC_s \cos(\beta - \alpha_i - \varphi) + V_sC_s \sin \varphi\), \(b_2 = O_iC_s \cos(\beta - \alpha_i - \varphi) - V_sC_s \sin \varphi\) for a recess one. Considering that \(V_sC_s = R_s \sin \alpha_i / \cos(\beta + \alpha_i)\), \(O_iC_s = \tau O_sC_s\) using Eqs. (11) and (18) and carrying out some calculations, the values of \(b_1\) and \(b_2\) are found to be given by

\[
\begin{align*}
\beta_1 &= \frac{R_s [\cos(\beta + \alpha_i - \alpha_{1s}) \cos(\beta + \alpha_i + \varphi) + \sin \alpha_i \sin \varphi]}{\cos(\beta + \alpha_i)} \quad (28) \\
\beta_2 &= \frac{R_s [\cos(\beta + \alpha_i - \alpha_{1s}) \cos(\beta + \alpha_i + \varphi) - \sin \alpha_i \sin \varphi]}{\cos(\beta + \alpha_i)} \quad (29)
\end{align*}
\]

during the access, and

\[
\begin{align*}
\beta_i &= \frac{R_s [\cos(\beta + \alpha_i - \alpha_{1s}) \cos(\beta + \alpha_i - \varphi) - \sin \alpha_i \sin \varphi]}{\cos(\beta - \alpha_i)} \quad (30) \\
\beta_i &= \frac{R_s [\cos(\beta + \alpha_i - \alpha_{1s}) \cos(\beta + \alpha_i + \varphi)]}{\tau + \sin \alpha_i \sin \varphi} \quad (31)
\end{align*}
\]

during the recess.

The efficiency \(\eta_f\) due to the only sliding friction is obtainable as the product of the speed ratio \(\tau\) and the arm ratio \(b_2 / b_1\):

\[
\eta_f = \frac{\cos(\beta + \alpha_i - \alpha_{1r}) \cos(\beta + \alpha_i + \varphi) - \sin \alpha_i \sin \varphi}{\cos(\beta - \alpha_i - \alpha_{1r}) \cos(\beta + \alpha_i + \varphi) + \sin \alpha_i \sin \varphi} \quad (32)
\]

\[
\eta_f = \frac{\cos(\beta + \alpha_i - \alpha_{1r}) \cos(\beta + \alpha_i + \varphi) + \sin \alpha_i \sin \varphi}{\cos(\beta - \alpha_i - \alpha_{1r}) \cos(\beta + \alpha_i - \varphi) - \sin \alpha_i \sin \varphi} \quad (33)
\]
The sliding friction energy loss during one complete meshing of two conjugate teeth can be calculated by integration:

$$L_f = \int_{\Delta \alpha_s} (1 - \eta) M_i d\alpha_i$$  \hspace{1cm} (34)$$

where, in the case of two partial contact sub-regions, the integration must be extended to both of them. The torque applied to the driven gearwheel is $M_2$, and is variable owing to the variability of the arms $b_1$ and $b_2$. Furthermore, this torque differs from the output resistant torque due to the secondary shaft inertia and the variability of $\omega_s$.

As regards the impact losses, the previous assumptions of constancy of the driving speed $\omega_1$ and inelastic impacts lead to the conclusion that, for every tooth collision, the work $\frac{1}{2}J_2\alpha_i^2(\alpha_{1i}^2 - \alpha_{2i}^2)$ must be provided by the driving wheel, where $J_2$ indicates the moment of inertia of the driven wheel and all the masses connected to the driven shaft, while the speed ratios $\tau$ refer to the instants immediately after and before the impact, to be calculated as in the previous section. In the case of two partial contact sub-region or of one single region astride the matching position, two impact works must be taken into account, because of the access, recess and matching impact. Such impact works have to be added to the sliding friction work, Eq. (34), to get the total lost energy due to the tooth coupling during one single mesh.

V. Results

Figures 3 to 5 show the numerical results arising from the above formulation for three tooth ratios $z_2/z_1$ and two values of the allowance factor $a = D/(R_1 + R_2 - h)$.

Figures 3 a and b refer to a speed down case. The active fraction of the diagram is confined in the recess region and its width along the abscissa axis is $360^\circ/z_1$. The concavity of the diagram $\alpha_\delta(\alpha_i)$ is slight and upward directed, so that the admissible active points are confined under the bottom chord of width $360^\circ/z_1$ and slope $z_1/z_2$. The speed ratio decreases along the direction of the meshing evolution, i.e. from right to left.

Close to the matching position, $\alpha_{1m}$, the distance of the contact point from the relative rotation center $C$ is very small, which justifies the high sliding efficiency near this position. A coefficient of friction $f = 0.2$ was chosen in all the examples, assuming bronze gearwheels and some sort of rudimentary lubrication, which compensates for the necessarily crude profile finishing somehow.

Comparing Fig. 3 a with Fig. 3 b, which refers to a larger backlash, it is possible to observe a significant reduction of the ideal meshing width, but no remarkable changes in the real mesh. Therefore, rather large allowances can be adopted without any important worsening of the mesh conditions, to the advantage of the prevention of possible transmission stops due to the tooth locking.

![Diagram](image-url)
Figures 4 a and b refer to a speed up case and similar comments can be made as for Figs. 3. The main difference is that the real mesh phase develops inside the access region, but tends to develop astride the matching position on increasing the backlash. This last feature is clearly typical of all cases.

Figures 5 a and b refer to a unitary speed ratio. For small allowance, two partial meshing regions can be observed, the one in the access region and the other in the recess one. The sum of the widths of these two sub-regions along the \( \alpha \) axis is equal to 360°/\( z_1 \) and is also equal to the distance between them. Therefore, indicating with the numbering, \( j \rightarrow j, \ j + 1 \), the sequence of three successive couples of conjugate teeth and starting the analysis of the engagement with the access sub-phase of the couple \( j \) from the right of the diagram towards the central position, the couple \( j - 1 \) begins the engagement of the other sub-phase, in the recess region, immediately after the conclusion of the access sub-phase of \( j \), and keeps on until this recess sub-phase is entirely covered. Then, the couple \( j + 1 \) starts a new access sub-phase from the right end and the whole process begins again. This sequence is permitted by the circumstance that the distance between the two extremes endpoints, on the right of the access contact and on the left of the recess contact, is exactly equal to 2×360°/\( z_1 \).

On increasing the allowance, the left sub-region width decreases to the advantage of the right one, their total width and their mutual distance remaining unchanged, until it vanishes and leaves only one single contact region. On increasing the allowance further, this unique phase moves to the left, until straddling the matching configuration \( \alpha_{sw} \) and lying in part in the access region and in part in the recess one. This is clearly visible in the case of Fig. 5 b, where the sliding efficiency reaches its highest average value and the impact velocity jump \( v_{a1} \) is very low. For this configuration, i.e. speed ratio 1:1 and rather large backlash, we get thus the best conditions as regards the energy losses. Clearly the gearwheel allowance cannot be increased too much because, if the theoretical contact region becomes smaller of the real one, we have standstill period of the driven wheel and important collisions at the new motion start.

VI. Conclusion

The extraordinary advancement degree of the Hellenistic science in conceiving planetary gear systems is somehow overshadowed by the technology primitiveness of that time. Nevertheless, an appreciable level of functionality can be detected by an accurate analysis. The gearing behavior was certainly characterized by a sensible rattling due to the tooth collisions consequent to the variability of the speed ratio. The energy losses were of course more relevant in comparison with the present gear systems, but, guessing that a sort of rudimentary greasing or oiling of the contact might have been applied even in the antiquity, the friction losses due to the rough technology in the tooth construction might have been compensated in part by the lubrication. Thus, energy losses of the order of 10% or less may be reasonably conjectured.

References